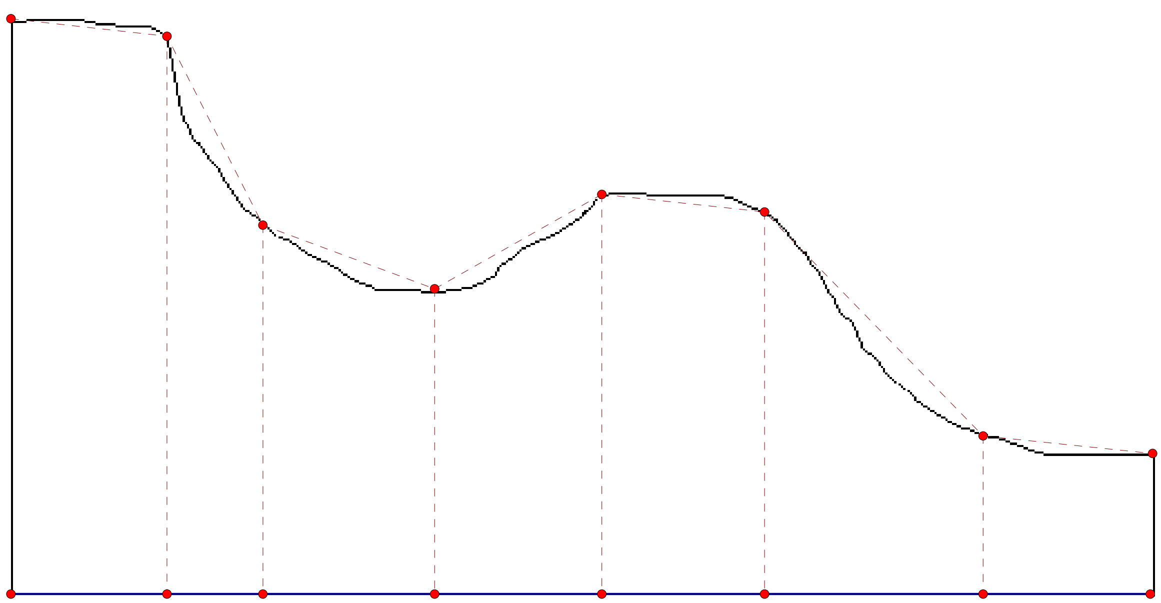
# GPS-UTM Module 9: *Can You Take It To The Limit?*

**Topics Covered:** Area under a curve, sigma notation, limits

**Required Background Material:** GPS-UTM Module 8, experience with trapezoids

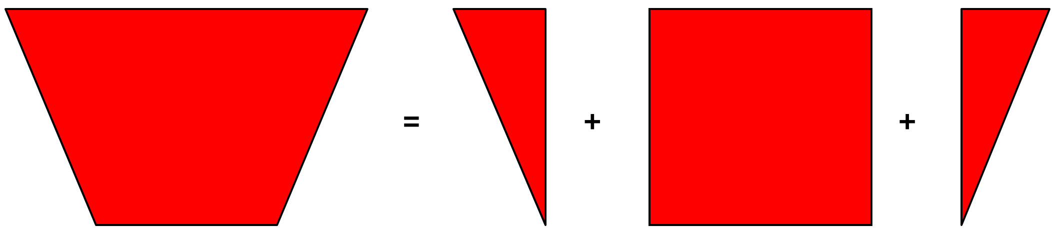
**Introduction**

As with Module 8, this unit will expand on the techniques introduced in Module 6, *How Much Land Does Grandfather Own?* If grandfather's property was bordered by a lake or stream, the Module 6 approach was to approximate the region by dividing it into rectangles and triangles. You might also have divided the property into trapezoids, as shown in brown below.



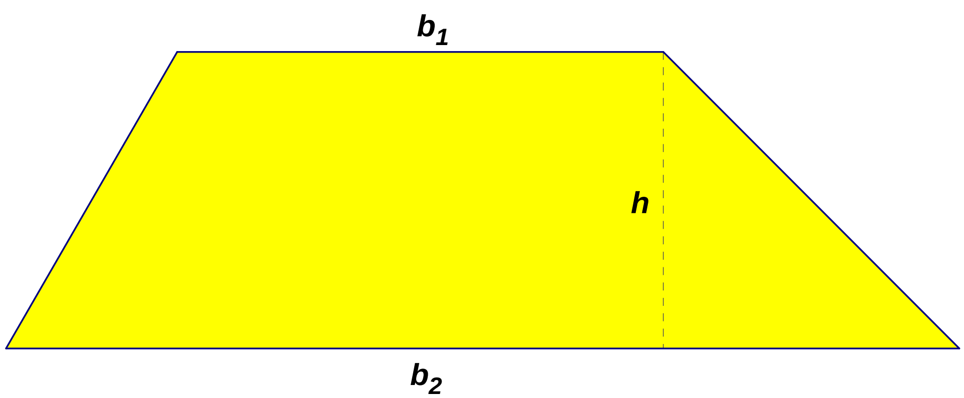
There are several problems with this technique. One problem is that you would need to find the GPS coordinates of the many points across the bottom line, in addition to the points on top. You would also have to be sure that the new shapes were trapezoids (quadrilaterals with one pair of parallel sides). And you would need to know how to find the area of a trapezoid.

You can avoid remembering the formula for the area of a trapezoid, if you just divide all of the trapezoids into rectangles and triangles. (See diagram below.)



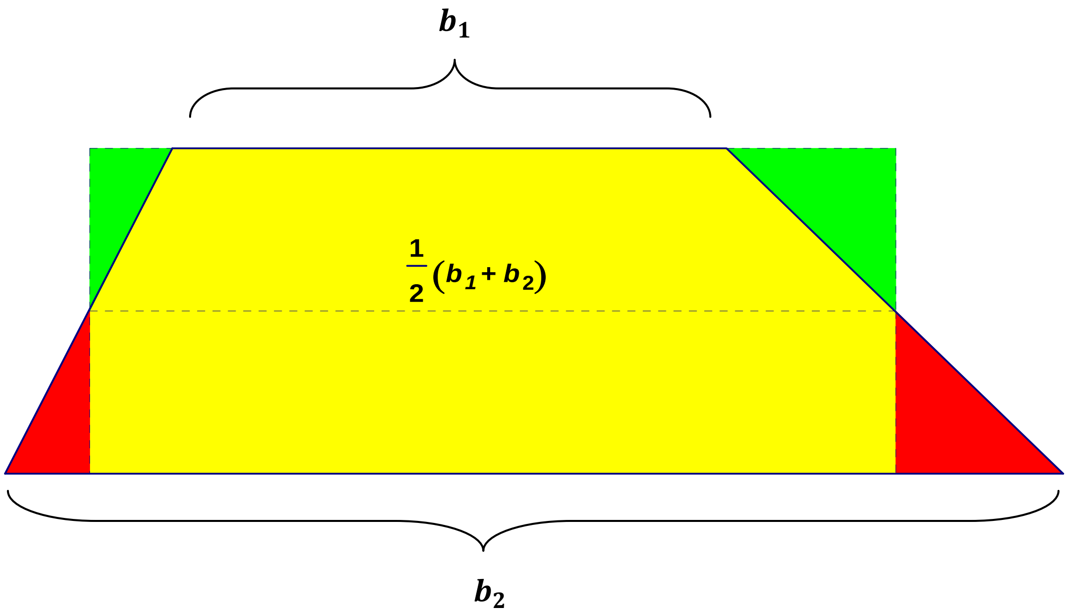

That is not a particularly efficient way to find the area of a trapezoid, because it involves finding GPS coordinates of two more points.

The two parallel sides of a trapezoid are called the bases. They are designated by $b_1$ and $b_2$ in the diagram below. The altitude is designated by $h$.




$$\textbf{Area} = \frac{h}{2}(b_1 + b_2)$$

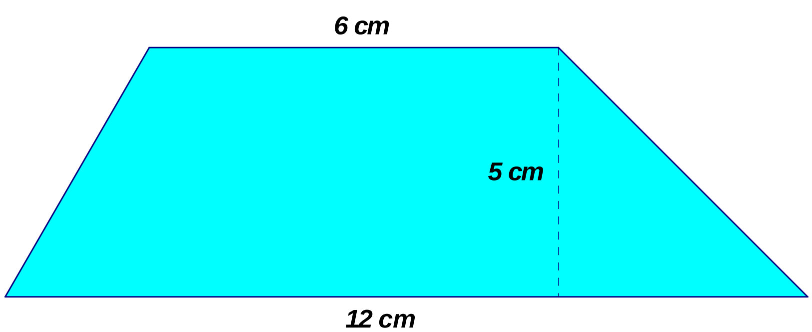
If you "cut and paste" parts of the above figure, it is easy to see where the area formula comes from. In the diagram on the next page, if the red areas are removed and the green areas are pasted in, the figure becomes a rectangle with height $h$ and length equal to the average of $b_1$ and $b_2$.




**A Trapezoid and Its Equivalent Rectangle**

**Problem 1**

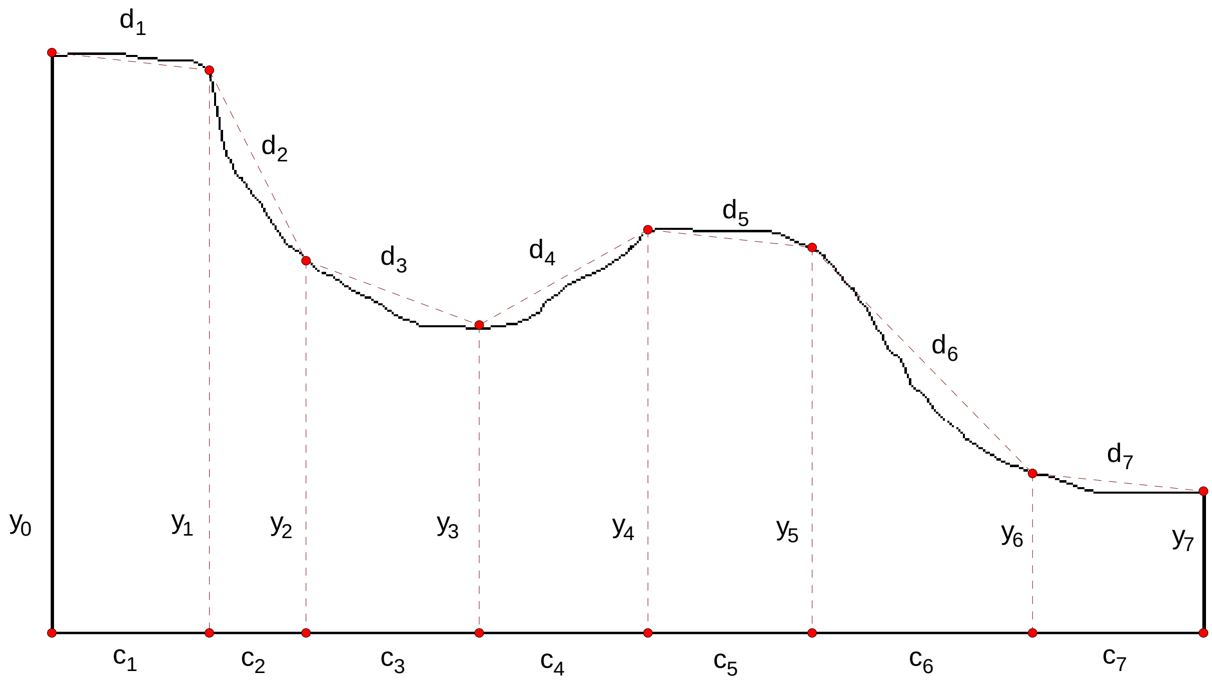
Find the area of the trapezoid in the following figure.




Answer: ________________

**Problem 2**

Returning to the figure on p.1 of this module, you should now be able to approximate the area of the region. Use the measurements shown in the redrawn figure on the next page.




| | | |
|---|---|---|
| $d_1$ = 1.86 cm | $c_1$ = 1.86 cm | $y_0$ = 6.93 cm |
| $d_2$ = 2.54 cm | $c_2$ = 1.13 cm | $y_1$ = 6.72 cm |
| $d_3$ = 2.18 cm | $c_3$ = 2.05 cm | $y_2$ = 4.45 cm |
| $d_4$ = 2.29 cm | $c_4$ = 1.96 cm | $y_3$ = 3.68 cm |
| $d_5$ = 1.94 cm | $c_5$ = 1.94 cm | $y_4$ = 4.82 cm |
| $d_6$ = 3.74 cm | $c_6$ = 2.60 cm | $y_5$ = 4.60 cm |
| $d_7$ = 2.02 cm | $c_7$ = 2.01 cm | $y_6$ = 1.91 cm |
| | | $y_7$ = 1.69 cm |

# Answer: ________________

**Sigma Notation**

Calculating the answer to Problem 2 is somewhat tedious, but it would take much longer if you had to measure all of the 22 distance values. Fortunately there are several shortcuts that have been developed through the years. Notice that

$$\textit{Total Area} =$$

$$c_1\left(\frac{y_0+y_1}{2}\right)+c_2\left(\frac{y_1+y_2}{2}\right)+c_3\left(\frac{y_2+y_3}{2}\right)+c_4\left(\frac{y_3+y_4}{2}\right)+c_5\left(\frac{y_4+y_5}{2}\right)+c_6\left(\frac{y_5+y_6}{2}\right)+c_7\left(\frac{y_6+y_7}{2}\right).$$

Since the $c_i{}'$s are perpendicular to the $y_i{}'$s, they can be thought of as the altitudes of the trapezoids, while the $y_i{}'$s form the bases. We can thus ignore the values of the $d_i{}'$s.

The formula for the total area can be abbreviated by the use of *sigma notation*, a shortcut way to write repeated summation. Here is the rewritten formula.

$$Total\ Area = \sum_{n=1}^{7} c_n \left(\frac{y_{n-1} + y_n}{2}\right)$$

Notice how the notation makes it easy to tell that the total number of terms is seven. It also emphasizes the similar pattern for each term. Each term is found by substituting a different value for *n*, starting with the integer 1 and ending with the integer 7. Best of all, the sigma notation form is more compact than the first way we did it.

**Problem 3**

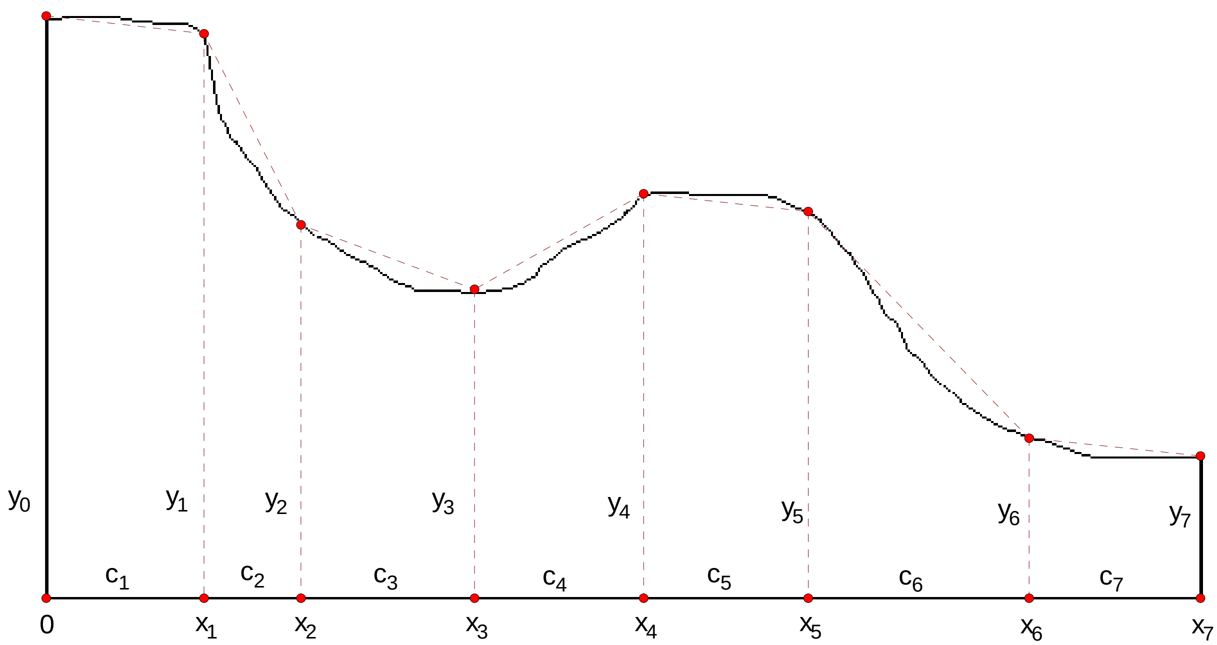
Write out the individual terms that are represented by the following formula.

$$\sum_{n=1}^{5} \frac{x^n}{2n-1}$$

Answer: ______________________________________________

**The Trapezoidal Rule**

It is time consuming to compute all of the values for the $c_i{}'$s and $y_i{}'$s whenever we need to find the area of a region with a curved boundary. The curve fitting from Module 8 can make this easier. If we adjust the x-values so that the smallest ones are zero, we can (as in Problem 6, from Module 8) label our area graph in the following manner.




The diagram is now arranged as if it was drawn on a Cartesian coordinate system, with the origin in the lower left hand corner. The $x_i$'s and $y_i$'s represent horizontal and vertical distances from the origin. Also,

$$c_1 = x_1$$

$$c_2 = x_2 - x_1$$

$$c_3 = x_3 - x_2$$

$$c_4 = x_4 - x_3$$

$$c_5 = x_5 - x_4$$

$$c_6 = x_6 - x_5$$

$$c_7 = x_7 - x_6$$

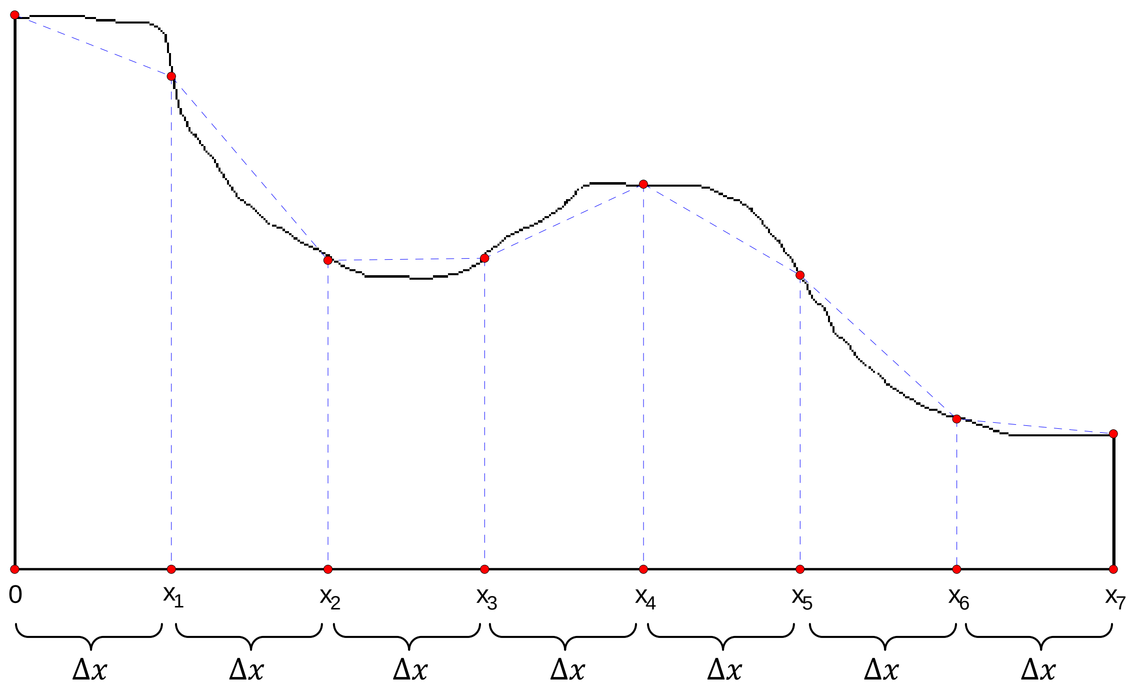
In other words, if $x_0 = 0$, $c_n = x_n - x_{n-1}$ for $n = 1, 2, 3, 4, 5, 6, 7$.

Furthermore, if we can approximate the curve with a polynomial $f(x)$, then $y_n \approx f(x_n)$ for $n = 1, 2, 3, 4, 5, 6, 7$. Our formula for area now looks like this:

$$Total\ Area = \sum_{n=1}^{7} (x_n - x_{n-1}) \left( \frac{f(x_{n-1}) + f(x_n)}{2} \right)$$

So, we don't need the $c_i$'s, and we don't even need the $y_i$'s (after we have done the curve fitting). All we need are the $x_i$'s.

One of the "magic" things about curve fitting is that, when finding the area, we don't need to choose the same $x_i$'s that we used to do the curve fit. We can use any $x_i$'s. To make it really easy, we can choose $x_i$'s that make all of the $c_i$'s the same size. This choice of $x_i$'s is called a *normal partition* of the base line. We call the width of the intervals $\Delta x$ (read "delta x"). In other words, $\Delta x = c_i = x_i - x_{i-1}$ for all values of *i.* Here is the graphical representation (in blue).




**A Uniform Partition of the Baseline**

The formula for area now becomes

$$Total\ Area = \sum_{n=1}^{7} \Delta x \left(\frac{f(x_{n-1}) + f(x_n)}{2}\right)$$

$$= \frac{\Delta x}{2}[(f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + (f(x_2) + f(x_3)) + (f(x_3) + f(x_4)) + (f(x_4) + f(x_5)) + (f(x_5) + f(x_6)) + (f(x_6) + f(x_7))]$$

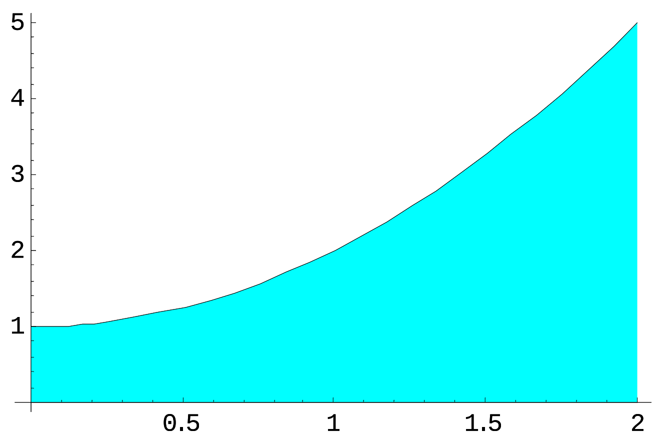
$$= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + 2f(x_6) + f(x_7)]$$

This is an example of a famous formula called **The Trapezoidal Rule**. In general, for $N$ trapezoids and a uniform partition of the baseline,

$$\boldsymbol{Total\ Area} = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(x_N)].$$

**Problem 4**

For a uniform baseline partition of $N$ intervals between $x = a$ and $x = b$, $\Delta x = \frac{b-a}{N}$. Then $x_0 = a_1$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, $\cdots$, $x_k = a + k\Delta x$, $\cdots, x_N = b$. (Where $k$ represents a number between 1 and $N$.) Use this formula, and the Trapezoidal Rule, to estimate the area under the curve of $f(x) = x^2 + 1$ for $a = 0, b = 2$, and $N = 4$.




**The Area Under $f(x) = x^2 + 1$**

Answer: ________________

**Problem 5**

Repeat Problem 4, using $N = 8$.

Answer: ________________

**Problem 6**

Compare your answers to Problems 4 and 5. The actual value of the area should be $\frac{14}{3}$, or approximately 4.66667. What do you think will happen if you use $N = 16$?

**Problem 7 (Opt.)**

In a calculus book, or on the internet, look up the definition of the *definite integral*. The symbol is $\int_a^b f(x)dx$. How does it compare to the area under the curve? What is the definition of the mathematical term *limit*?

**Problem 8**

Use the polynomial fit function that you found in Module 8, Problem 6, to estimate the area of a piece of land with a curved border between two different *x* values. Be sure to adjust the numbers, as shown in Module 8, so that the smallest *y* value is zero. You will then have an estimate for a region similar to the one shown in yellow below. By adding this to the area of the remaining plot of land (shown in green below), you can estimate the area of the entire region.

**An Estimate for the Area of an Irregularly Shaped Region**